Our next goal is to show that $\xi(s)$, $L(f,s)$ for $f \in M_k(\Gamma)$ can be analytically continued to the whole s-plane with possibly finitely many poles.

We start with the prototype $\xi(s)$ but first need to introduce 2 other functions.

Def: **Euler Gamma function**

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}, \quad \mathrm{Re}\, s > 0.$$

We collect the analytic properties of Γ in the following thm (for proofs see Freitag Busam complex analysis)

Thm 4.5 ① $\Gamma(s)$ has A.C to the whole complex plane with simple poles at $s = 0, -1, -2, \ldots$ w/ residue $(-1)^n/n!$ at $s=-n$

② It satisfies the functional eqn

$$\Gamma(s+1) = s\Gamma(s), \quad \Gamma(1) = 1$$

Hence $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}^+$

③ $\Gamma(s)\Gamma(1-s) = \pi / \sin \pi s$

④ $\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right) = \sqrt{\pi}\, 2^{1-s}\, \Gamma(s)$ (duplic. formula)

⑤ $1/\Gamma(s)$ is an entire func. of s.

(6) (real) Stirling's formula

As $x \to \infty$, $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$

(ie $\dfrac{\Gamma(x+1)}{\sqrt{2\pi x}\left(\frac{x}{e}\right)^x} \to 1$ as $x \to \infty$)

$\Gamma(x)$ is a special case of an integral transform, the so called Mellin transform.

Defn Given a function $f: \mathbb{R}^+ \to \mathbb{C}$ s.t

$$f(y)\, y^{s-1} \in L^1(\mathbb{R}^+) \quad \text{for } \alpha < \operatorname{Re} s < \beta$$

The Mellin transform of f is defined as

$$\mathcal{M}(f)(s) := \int_0^\infty f(y)\, y^{s-1}\, dy \quad , \alpha < \operatorname{Re} s < \beta$$

eg $\Gamma(s) = \mathcal{M}(e^{-y})(s) \qquad 0 < \operatorname{Re} s < \infty$.

The 2nd function we need to study the analytic behaviour of $\zeta(s)$ is the theta function

Theta functions

Defn Theta functions with characteristic.
Let $u, v \in \mathbb{C}$, $\tau \in \mathbb{H}$ then

$$\vartheta_{u,v}(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i (n+u)^2\tau + 2\pi i (n+u)v}.$$

is a theta series with characteristic (u,v).
In the special case $u=v=0$ we have

$$\vartheta_{0,0}(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau} = 1+2\sum_{n=1}^{\infty} e^{\pi i n^2\tau}.$$

is a Theta-Nullwert.

Lemma 4.6 The theta series $\vartheta_{u,v}(\tau)$ converges abs and uniformly on compacta. In particular $\vartheta_{u,v}(\tau)$ is holomorphic as a function of $u, v \in \mathbb{C}$ and $\tau \in \mathbb{H}$.

Proof. Let $K \subset \mathbb{C}\times\mathbb{C}\times\mathbb{H}$ compact.
Then $\exists$ $c>0$ s.t.

$$|u| \leq c \quad |v| \leq c \quad |\tau| \leq c \quad \text{and}$$

$$\operatorname{Im}\tau \geq \frac{1}{c} \quad \forall (u,v,\tau) \in K.$$

If $(u,v,\tau) \in K$ with $u = u_1 + iu_2$ $v = v_1 + iv_2$
$\tau = x+iy$ then

$$R := \operatorname{Re}\left(\pi i (n+u)^2 \tau + \pi 2 i (n+u) v\right)$$

$$= \pi\left(-y(n+u_1)^2 + y u_2^2 - 2 x u_2 (n+u_1) - 2 v_2 (n+u_1) - 2 u_2 v_1\right)$$

$$\leq \pi\left(-\frac{1}{c}(n+u_1)^2 + c^3 + 2c(c+1)|n+u_1| + 2c^2\right)$$

$\exists\, N \in \mathbb{N}$ s.t. $\forall |n| > N$

$$\frac{1}{2}|n| \leq |n+u_1| \leq 2|n|$$

and $$R \leq -\frac{\pi}{8c} n^2 < -\frac{\pi}{8c}|n|$$

Hence $$\left|\Theta_{u,v}(\tau)\right| \leq \sum_{n\in\mathbb{Z}} \left| e^{\pi i (n+u)^2 \tau + 2\pi i (n+u) v}\right|$$

$$\leq 1 + 2\sum_{n\geq 0} e^{R} \leq 1 + 2\sum_{n\geq 0}\left(e^{-\pi/8c}\right)^n$$

And the geometric series $\sum\left(e^{-\pi/8c}\right)^n$ is a Majorant and hence $\Theta_{u,v}(\tau)$ conv. unif. on K

Weierstrass' thm on uniform convergence (Ahlfors: Complex Analysis Chapter V Section 1-1) implies $\Theta_{u,v}(\tau)$ is holom. as a func of u, v, τ.

For our purposes 4-1 $\boxed{4}$

We only ~~really~~ only need $\theta_{0,0}(\tau) = \theta(\tau)$

Its relation to $\zeta(s)$ comes from the simple identity

$$\int_0^\infty e^{-at} t^s \frac{dt}{t} = \Gamma(s) a^{-s} \qquad a > 0.$$

with $a = n^2 \pi$ and $s \to s/2$

$$\int_0^\infty e^{-n^2 \pi t} t^{s/2} \frac{dt}{t} = \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} n^{-s}$$

Summing over all $n \geq 1$, interchanging the sum and the integral (using Fubini / Tonelli)

$$\zeta(s) \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} = \int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 t}\right) t^{s/2} \frac{dt}{t}$$

$$= \frac{1}{2} \int_0^\infty \left(\theta(it) - 1\right) t^{s/2} \frac{dt}{t}.$$

where $\theta(it) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t}$, $t > 0$.

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-s/2} = \frac{1}{2}\int_0^{i\infty}\left(\Theta(it)-1\right)t^{s/2}\frac{dt}{t}$$

gives an integral representation for $\zeta(s)$ and we'll use this integral repr. to analytically continue $\zeta(s)$

To this end we need transformation of the theta func.

Thm 4.7 $\Theta_{u,v}(\tau)$ satisfies the transf. law and $\Theta_{-v,u}\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\, e^{-2\pi i uv}\,\Theta_{u,v}(\tau)$

where $\sqrt{\ }$ is given by the principal branch (ie the cut is -ve real axis) argument $\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Proof Since $\Theta_{u,v}(\tau)$ is holomorphic in u, v, τ, Due to uniqueness theorem for holomorphic functions it suffices to prove the identity for $\tau = iy$.

ie w.t.s $\Theta_{-v,u}\left(\frac{i}{y}\right) = \sqrt{y}\, e^{-2\pi i uv}\,\Theta_{u,v}(iy)$.

let $u \in \mathbb{C}$, $y > 0$ and consider the function $\phi : \mathbb{C} \to \mathbb{C}$

$v \to \phi(v) := \Theta_{-v,u}(i/y)$

Fe

$$\phi(v) = \sum_{n\in\mathbb{Z}} e^{-\pi\frac{(n-v)^2}{y} + 2\pi i (n-v)u}$$

Clearly $\phi(v+1) = \phi(v)$ and hence ϕ has a Fourier expansion

$$\phi(v) = \sum_{m\in\mathbb{Z}} a_m e^{2\pi i m v} \quad \text{with}$$

$$a_m = \int_0^1 \phi(v) e^{-2\pi i m v} dv$$

$$= \int_0^1 \sum_{n\in\mathbb{Z}} e^{-\pi\frac{(n-v)^2}{y}} \cdot e^{+2\pi i(n-v)u} \cdot e^{-2\pi i m(v-n)} dv$$

$e^{2\pi i m n} = 1$

$v - n = v$

$$\sum_{n\in\mathbb{Z}} \int_n^{n+1} e^{-\pi v^2/y} \cdot e^{-2\pi i v(m+u)} dv$$

$$= \int_{-\infty}^{\infty} e^{-\pi v^2/y - 2\pi i v(m+u)} dv$$

$$= e^{-\pi (m+u)^2 y} \int_{-\infty}^{\infty} e^{-\pi\left(\frac{v}{\sqrt{y}} + i\sqrt{y}(m+u)\right)^2} dv$$

complete the square

$$\underset{t = v/\sqrt{y}}{=} e^{-\pi(m+u)^2 y} \sqrt{y} \underbrace{\int_{-\infty}^{\infty} e^{-\pi(t + i\sqrt{y}(m+u))^2} dt}_{=1}$$

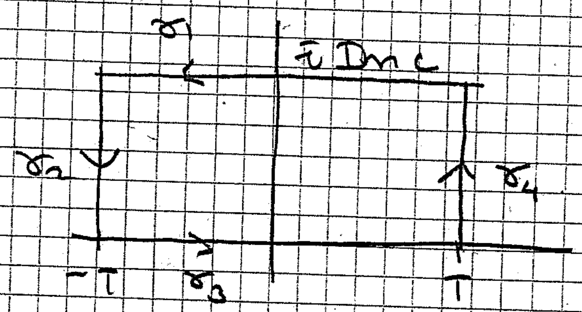
$\sqrt{y}\, dt = dv$

Now we have

Claim
$$\int_{-\infty}^{\infty} e^{-\pi(t-c)^2}\,dt = \int_{-\infty}^{\infty} e^{-\pi t^2}\,dt = \frac{1}{\sqrt{\pi}}\,\Gamma\!\left(\frac{1}{2}\right) =$$

for any $c \in \mathbb{C}$.

Proof Let $f(s) := e^{-\pi(s-c)^2}$. Let γ be the path $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$



Cauchy's formula gives

$$\int_\gamma f(s)\,ds = 0$$

Check that ① $-\int_{\gamma_1} f\,ds = \int_{-T}^{T} e^{-\pi(t-\operatorname{Re}c)^2}\,dt$

$$\underset{t \to t+\operatorname{Re}c}{=} \int_{-T+\operatorname{Re}c}^{T+\operatorname{Re}c} e^{-\pi t^2}\,dt \xrightarrow[T\to\infty]{} \int_{-\infty}^{\infty} e^{-\pi t^2}\,dt$$

② $$\int_{\gamma_3} = \int_{-T}^{T} e^{-\pi(t-c)^2}\,dt \xrightarrow[T\to\infty]{} \int_{-\infty}^{\infty} e^{-\pi(t-c)^2}\,dt$$

③ $$\int_{\gamma_2} f(s)\,ds = -i\int_0^{\operatorname{Im}c} e^{-\pi(-T+i(\operatorname{Im}c-t)-c)^2}\,dt \xrightarrow[T\to\infty]{} 0$$

④ $$\int_{\gamma_4} f(s)\,ds = i\int_0^{\operatorname{Im}c} e^{-\pi(T+it-c)^2}\,dt \xrightarrow[T\to\infty]{} 0$$

Hence $\int_{-\infty}^{\infty} e^{-\pi(t-c)^2}\,dt = \int_{-\infty}^{\infty} e^{-\pi t^2}\,dt$

Exercise: $\int_{-\infty}^{\infty} e^{-\pi t^2}\,dt = 1$

Hence we've shown $a_m = e^{-\pi(m+u)^2 y} \cdot \sqrt{y}$

and $\theta_{-v,u}\left(\frac{i}{y}\right) = \sum_{m\in\mathbb{Z}} a_m e^{2\pi i m v}$

$$\sqrt{y} \sum_{m\in\mathbb{Z}} e^{-\pi(m+u)^2 y} \cdot e^{2\pi i m v}$$

$$= \sqrt{y}\, e^{-2\pi i u v} \sum_{m\in\mathbb{Z}} e^{-\pi(m+u)^2 y} \cdot e^{2\pi i (m+u) v}$$

$$= \sqrt{y}\, e^{-2\pi i u v}\, \theta_{u,v}(iy) \quad \text{as wanted}$$

$\square$.

As a corollary we get for $\theta_{0,0}(\tau) = \theta(\tau)$

Cor 4.8 a) $\theta(\tau+2) = \theta(\tau)$

and b) $\theta\left(-\frac{1}{\tau}\right) = \sqrt{\frac{\tau}{i}}\, \theta(\tau)$

Rk. Note these transformations correspond to $\tau \to S\tau$ and $\tau \to T^2\tau$ and they generate the so called theta subgroup Γ_θ

Rmk Another way to prove Thm 4.7 and in particular (Cor 4.7 b) is to use an important tool from Fourier Analysis

Poisson Summation.

The Fourier transform on $\mathbb{R}^1$ is the operator on Lebesgue integrable functions given by

$$\hat{f}(y) := \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i xy}\, dx.$$

If f is continuous and $\hat{f}$ is integrable then Fourier inversion says that

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(y)\, e^{+2\pi i xy}\, dy$$

For our purposes a good class of integrable functions is the space of Schwarz functions

$$\mathcal{S}(\mathbb{R}) = \Big\{ f \in C^\infty(\mathbb{R}) \;\Big|\; p(t)\frac{d^k f}{dt^k} \text{ is bounded for each } k \geq 0,\ k \in \mathbb{Z} \text{ and polynomial } p(t) \Big\}$$

Then a theorem from Fourier analysis says that Fourier transform carries $\mathcal{S}(\mathbb{R})$ one to one onto $\mathcal{S}(\mathbb{R})$.

Fourier transform satisfies the following properties ① For $a \in \mathbb{R}$, let $g(x) := f(x+a)$

then $\hat{g}(y) = e^{2\pi i a y} \hat{f}(y)$

② $a \in \mathbb{R}$, $g(x) := e^{2\pi i a x} f(x)$ then

$\hat{g}(y) = \hat{f}(y-a)$

③ $b>0$, $g(x) := f(bx)$ then

$\hat{g}(y) = \frac{1}{b} \hat{f}\left(\frac{y}{b}\right)$

The following theorem is a very powerful tool

Thm 4.9 (Poisson summation)

Let $f \in \mathcal{S}(\mathbb{R})$ then

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}.$$

Cor 4.10 If $f \in \mathcal{S}(\mathbb{R})$ then

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n)$$

One can now prove Cor 4.7 also as follows

Exercise ① Let $f(x) = e^{-\pi x^2}$ Then $\hat{f}(y) = e^{-\pi y^2}$

② Let $g(x) = e^{-\pi t x^2}$, $t>0$. Then $\hat{g}(y) = \sqrt{t}\, e^{-\pi y^2/t}$

③ Use Thm 4.9 (P.S) to get

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = \sqrt{t} \sum e^{-\pi n^2/t}.$$